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Homogenization of a Hamilton–Jacobi equation associated with the geometric motion of an interface

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This paper studies the overall evolution of fronts propagating with a normal velocity that depends on position, $v_n = f(x)$, where f is rapidly oscillating and periodic. A level-set formulation is used to rewrite this problem as the periodic homogenization of a Hamilton–Jacobi equation. The paper presents a series of variational characterization (formulae) of the effective Hamiltonian or effective normal velocity. It also examines the situation when f changes sign.

1. Introduction

The present paper studies the evolution of interfaces or fronts propagating with a normal velocity that depends on position. Consider, for example, a free boundary propagating with the normal velocity

$$v_n = f(x) = \sigma - \tilde{\sigma}(x)$$

inside a heterogeneous body occupying a region $\Omega \subset \mathbb{R}^N$, where σ is a constant and $\tilde{\sigma} : \Omega \to \mathbb{R}$ is a spatially rapidly oscillating function. This problem is motivated by the study of phase transformations [4,8], where σ is the applied load promoting the phase transformation and $\tilde{\sigma}$ is the resistance of the medium to changing phase. If the length-scale of the oscillations in $\tilde{\sigma}$ are small compared to typical length-scales in Ω, then the detailed evolution of the front is rather complicated. However, it is often possible to define an overall or effective front which averages over the complicated details. This paper seeks to find the law that governs the overall evolution of the effective front when $\Omega = \mathbb{R}^N$ and $\tilde{\sigma}$ is periodic.

Similar problems arise in the calculation of first arrivals in seismic travel times (with a velocity that varies depending on the type of rock) and the development process in photolithography, where the resistive strength of the material is differentially altered through optical processes and the material is then exposed to an etching beam that removes the weaker material.

An efficient tool for studying such problems is the level-set formulation. If we assume that there exists a smooth function $h : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}$ such that our front coincides with its zero level set at all times, a simple calculation yields

$$n = \frac{\nabla h}{|\nabla h|}, \qquad v_n = -\frac{h_t}{|\nabla h|},$$

where n denotes the normal to the front and h_t denotes the derivative of h with respect to the time t. It follows that h satisfies the following Hamilton–Jacobi initial-value problem:

$$\begin{aligned} h_t + H(x, \nabla h) &= 0 && \text{in } \mathbb{R}^N \times [0, \infty), \\ h(x, 0) &= h_0(x) && \text{in } \mathbb{R}^N, \end{aligned}$$

with the Hamiltonian

$$H(x, p) = f(x)|p| \tag{1.1}$$

and the initial data $h_0 \in BUC(\mathbb{R}^N)$ (where BUC denotes the space of bounded and uniformly continuous functions) chosen such that its zero level set coincides with the initial position of the front. If $\mathcal{S}_0$ denotes the set of all points in the initial front, a common choice for h_0 is

$$h_0(x) = \min(d(x, \mathcal{S}_0), 1),$$

where $d(x, \mathcal{S}_0)$ is the distance from x to $\mathcal{S}_0$. This level-set approach is reasonable when f does not change sign, since the zero level set of the function h always has an empty interior by a result of Barles *et al.* [2, theorem 4.1].

Now, if the medium in which the front is propagating is periodic with unit cell $[0, \varepsilon]^N$, the corresponding Hamilton–Jacobi initial-value problem is

$$\left.\begin{aligned} h_t^\varepsilon + f\left(\frac{x}{\varepsilon}\right)|\nabla h^\varepsilon| &= 0 && \text{in } \mathbb{R}^N \times [0, \infty), \\ h^\varepsilon(x, 0) &= h_0(x) && \text{in } \mathbb{R}^N, \end{aligned}\right\} \tag{1.2}$$

with f continuous and periodic with period $Y_N = [0, 1]^N$. Our aim is to study the homogenization of this phenomenon, i.e. to capture its limit behaviour when the structure of the medium becomes infinitely fine ($\varepsilon \to 0$).

We say that a homogenized front exists and its level-set formulation is given by the Hamilton–Jacobi initial-value problem

$$\left.\begin{aligned} h_t + \bar{H}(\nabla h) &= 0 && \text{in } \mathbb{R}^N \times [0, \infty), \\ h(x, 0) &= h_0(x) && \text{in } \mathbb{R}^N \end{aligned}\right\} \tag{1.3}$$

if the viscosity solution h^ε of problem (1.2) converges uniformly on $\mathbb{R}^N \times [0, T)$ (for all $T < \infty$) to the viscosity solution of problem (1.3). We call $\bar{H}$ the effective or homogenized Hamiltonian. Furthermore, in analogy to $H(x, p) = f(x)|p|$, we write

$$\bar{H}(p) = \bar{v}_n(p)|p|$$

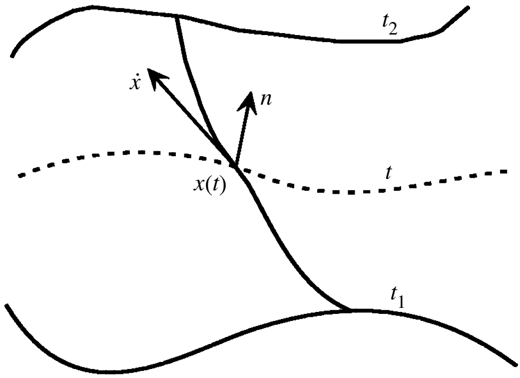

Figure 1. Snapshots of the propagating front at times t_1, t and t_2, and the trajectory of a point on the front.

and call $\bar{v}_n$ the effective normal velocity[1]. Lions *et al.* [12] and Evans [10] have shown the existence of the effective Hamiltonian under certain hypotheses, which, in our case, reduce to the strict positivity and Lipschitz continuity of f. They also discuss some properties of the effective Hamiltonian, but their results do not provide the means for calculating it.

The present paper addresses two issues: first the characterization of the effective Hamiltonian; and second a discussion of the situation when f changes sign. Our main results, theorems 3.4, 3.6 and 3.9, provide a series of variational characterizations or formulae for $\bar{H}$ when the Hamiltonian is of the form (1.1) with f strictly positive (or equivalently strictly negative). These formulae are based on the Lax-representation formula and the definition of the effective Hamiltonian in terms of this formula by E [9]. In other words, the formulae are based on paths and the time to traverse/distance traversed along them.

Consider a front propagating with normal velocity $v_n = f > 0$ during some interval of time (t_1, t_2), as shown in figure 1. Let $x(t)$ describe the trajectory or path followed by a point on this front. Since $v_n = \dot{x} \cdot n$, it follows that $|\dot{x}|(t) \geqslant f(x(t))$, i.e. the speed of the point along the path is greater than or equal to the normal velocity of the front. Therefore, the time T taken to traverse this same path with speed f is greater than or equal to the time $t_2 - t_1$ taken by the point following the front,

$$T = \int_{t_1}^{t_2} \frac{|\dot{x}(t)|}{f(x(t))} \, \mathrm{d}t \geqslant t_2 - t_1.$$

It follows that the average velocity following the front, $|x(t_2) - x(t_1)|/(t_2 - t_1)$, is greater than or equal to the average velocity while travelling at speed f, $|x(t_2) - x(t_1)|/T$. Furthermore, these are equal if and only if the trajectory is always normal to the front. Thus it is intuitively clear that the effective velocity can be obtained as the supremum over all paths connecting two traces of the front. Our results make this intuition precise.

Consider any path whose endpoints are separated by the vector d. Let the time to traverse this path with speed f be T. Then the projected average velocity in the

[1] We shall see that $\bar{v}_n(p)$ is homogeneous of degree zero and depends only on the direction of p.

direction p is

$$\frac{p}{|p|}\cdot\frac{d}{T}.$$

Theorem 3.4 states that the effective normal velocity $\bar{v}_n(p)$ in the direction p is obtained by taking the supremum of this projected average velocity amongst all paths of given distance $|d| = D$ and then passing to the limit $D \to \infty$. Theorem 3.6 states that the effective normal velocity $\bar{v}_n(p)$ in the direction p is obtained by taking the supremum of this projected average velocity over all paths that require time T to traverse and then passing to the limit $T \to \infty$. Finally, theorem 3.9 states that the effective normal velocity $\bar{v}_n(p)$ in the direction p is obtained by taking the supremum of this projected average velocity amongst all paths of given projected distance $p \cdot d = P$ and passing to the limit $P \to \infty$.

We then examine the issues that arise when f changes sign through three key examples. Example 5.1 shows that the front may assume an oscillatory shape with constant amplitude and frequency of order ε^{-1} as ε decreases[2]. In short, the front converges to a set with an interior. Example 5.2 shows that the front may become trapped in a smaller and smaller set as ε decreases and approaches a stationary front. Finally, example 5.3 shows that the front may shed isolated pieces as it propagates. We cannot study these problems with the level-set framework discussed above. In fact, we show that the homogenization of (1.2) makes sense only when f assumes strictly positive (or, equivalently, strictly negative) values (see remark 2.2). Yet, in many of these examples, one can identify a clear notion of effective front. And we find that our variational formulae give the velocity of this front. Since these formulae are based on the notion of paths, they make sense even when f changes sign. We conclude that when f changes sign, the effective front is decided by the nature of paths in regions where f is strictly positive or negative, and specifically whether the paths can percolate through regions where f is strictly positive or negative.

The paper is organized as follows. Section 2 collects the results on homogenization and also shows the necessity of strict positivity for using the level-set formulation and homogenization. Section 3 derives the variational characterization of the effective normal velocity. Section 4 uses these formulae to obtain some useful bounds. The discussion of the situation when f changes sign is contained in § 5. This section also contains other examples to illustrate the usefulness of our formulae. They also demonstrate that the homogenization of isotropic media may give birth to anisotropic ones.

2. Homogenization

We review the literature on the homogenization of (1.2) in this section. We begin with a formal asymptotic expansion (see [3] for a systematic presentation of such ansatz) by assuming that

$$h^{\varepsilon}(x,t) = h^0(x,t) + \varepsilon h^1\left(\frac{x}{\varepsilon}, t\right) + o(\varepsilon).$$

[2] This is allowed by the fact that the present model allows the front to develop infinite curvature. A model that avoids this by adding a multiple of $\varepsilon\kappa$ (where κ is the curvature) to the normal velocity law is studied in [7].

Plugging this into (1.2) and collecting terms of order 0, we find

$$h^0_t + f(y)|\nabla h^0 + \nabla_y h^1| = 0,$$

where $y = x/\varepsilon$. This is a partial differential equation for the corrector h^1 and its solvability condition provides a constraint between the partial derivatives of the average h^0,

$$h^0_t + \bar{H}(\nabla h^0) = 0,$$

with the effective Hamiltonian $\bar{H}$ is (uniquely) determined by the condition that a periodic solution v of

$$H(y, p + \nabla_y v) = \bar{H}(p) \tag{2.1}$$

exists.

This result has been made rigorous by Lions *et al.* [12] and by Evans [10].

THEOREM 2.1 (cf. [10]). *Let $H \in C(\mathbb{R}^N \times \mathbb{R}^N)$ be periodic in x and satisfy*

$$H(x,p) \to \infty \quad \text{as } |p| \to \infty \quad \text{uniformly for } x \in \mathbb{R}^N. \tag{2.2}$$

For each $p \in \mathbb{R}^N$, there exists a unique $\lambda \in \mathbb{R}$ (which we denote by $\bar{H}(p)$) such that there exists a periodic, viscosity solution $v \in C(\mathbb{R}^N)$ of

$$H(y, p + D_y v) = \lambda \quad \text{in } \mathbb{R}^N.$$

Furthermore, $\bar{H}$ is continuous in p.

For any $u_0 \in BUC(\mathbb{R}^N)$ (the space of bounded and uniformly continuous functions on $\mathbb{R}^N$), the solution u^ε of the Hamilton–Jacobi equation

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) &= 0 \qquad \text{in } \mathbb{R}^N \times [0,\infty), \\ u^\varepsilon(x,0) &= u_0(x) \quad \text{in } \mathbb{R}^N \end{aligned}$$

converges uniformly on $\mathbb{R}^N \times [0,T]$ $(\forall T < \infty)$ to the viscosity solution of

$$\begin{aligned} \frac{\partial u}{\partial t} + \bar{H}(Du) &= 0 \qquad \text{in } \mathbb{R}^N \times [0,\infty), \\ u(x,0) &= u_0(x) \quad \text{in } \mathbb{R}^N \end{aligned}$$

in $BUC(\mathbb{R}^N \times [0,T])$.

Moreover, they state that $\bar{H}$ is a continuous convex function and that $\bar{H}(p)$ goes uniformly to infinity as $|p| \to \infty$. They also derive some elementary bounds,

$$\inf_x f(x)|p| \leqslant \bar{H}(p) \leqslant \sup_x f(x)|p|. \tag{2.3}$$

While this proves the existence of $\bar{H}$, it does not provide a useful characterization. This can be obtained by looking at its dual. Define the Lagrangian associated to H by its Legendre dual

$$L(x,q) = \sup_{p \in \mathbb{R}^N} (q \cdot p - H(x,p)) = \begin{cases} 0 & \text{if } |q| \leqslant f(x), \\ \infty & \text{if } |q| > f(x). \end{cases} \tag{2.4}$$

In [9], E uses Γ-convergence techniques for the Lax-representation formula

$$h^\varepsilon(x,t) = \inf_y \biggl(h_0(y) + \inf_\xi \biggl(\int_0^t L\Bigl(\frac{\xi}{\varepsilon}, \dot{\xi}\Bigr)\,\mathrm{d}\tau \,\biggm|\, \xi(0) = y,\ \xi(t) = x,\ \xi \in W^{1,\infty}(0,t) \biggr) \biggr)$$

and proves that the solution of problem (1.2) converges uniformly to

$$\inf_y \biggl\{ h_0(y) + t\bar{L}\Bigl(\frac{x-y}{t}\Bigr) \biggr\},$$

where

$$\bar{L}(\lambda) = \liminf_{D\to\infty} \frac{1}{D} \inf_{\phi \in H_0^1(0,D)} \int_0^D L(\lambda t + \phi(t), \lambda + \dot{\phi}(t))\,\mathrm{d}t, \tag{2.5}$$

and that $\bar{L}$ is the dual of $\bar{H}$. This result thus characterizes $\bar{H}$ using a variational principle.

Note that the hypothesis (2.2) in the theorem is equivalent to the strict positivity of f in our case, i.e. when H is of the form (1.1). We now show that this is indeed necessary.

Remark 2.2. Assume that f and h_0 belong to $W^{1,\infty}(\mathbb{R}^N)$. To have a non-stationary homogenized level-set function, we need f to assume strictly positive values or strictly negative values.

Proof. We prove this result by contradiction. Assume that the function f is neither strictly positive nor strictly negative. By continuity and periodicity of f, we can find a point in any unit cell where it vanishes.

Since the homogenized front does move, there exists some compact set D in $\mathbb{R}^N$, with non-void interior, in which the function h changes sign. Without loss of generality, we may assume that $h_0(x) > 0$, $h(x,t) < 0$ for all x in D and for some strictly positive t (note that we may need to change f to $-f$ and h_0 to $-h_0$ for this to be true). By the uniform convergence of h^ε to h, there has to exist a positive ε_D such that, for any ε smaller than ε_D, $h^\varepsilon(x,t) < 0$ in D.

On the other hand, since the interior of D is not void, there exists some positive $\bar{\varepsilon}_D$ and some x_D in D such that $x_D + \varepsilon Y_n \subset D$ $\forall \varepsilon \leqslant \bar{\varepsilon}_D$.

Now fix ε to be the minimum of ε_D and $\bar{\varepsilon}_D$. By our assumption on f, we get a point $x \in D$ in which f_ε vanishes, where f_ε is defined by

$$f_\varepsilon(x) = f\Bigl(\frac{x}{\varepsilon}\Bigr).$$

Due to the uniform continuity of h^ε, there exists a positive real m such that

$$h^\varepsilon(x,0) > m, \quad h^\varepsilon(x,t) < -m \quad \forall x \in D. \tag{2.6}$$

Using the existence results developed by Lions in [11, ch. 9], we see that the viscosity solution of problem (1.2) belongs to $W^{1,\infty}(\mathbb{R}^N \times (0,t])$. Hence its spatial gradient is bounded,

$$|\nabla h^\varepsilon| \leqslant M \quad \text{a.e. in } D \times (0,t], \tag{2.7}$$

for some positive real M. Finally, using the continuity of f and the fact that it vanishes in x, we obtain a ball $B_\rho(x)$ contained in D for which

$$f(y) < \frac{2m}{tM} \quad \forall y \in B_\rho(x). \tag{2.8}$$

Now, inequalities (2.6) give us that

$$\int_0^t h_t(z,\theta)\,\mathrm{d}\theta < -2m \quad \forall z \in B_\rho(x),$$

which implies

$$\int_0^t f(z)|\nabla h^\varepsilon|\,\mathrm{d}\theta > 2m \quad \forall z \in B_\rho(x).$$

Combining this with (2.7) and (2.8), we obtain

$$\int_0^t \frac{2m}{tM} M\,\mathrm{d}\theta > 2m \quad \forall z \in B_\rho(x),$$

an obvious contradiction, since

$$\int_0^t \frac{2m}{tM} M\,\mathrm{d}\theta = 2m.$$

□

We also note that E's method using the Lax-representation formula also requires the strict positivity of f, since the Legendre dual is identically infinity otherwise.

Therefore, we assume in the next section that f is strictly positive everywhere (if f is strictly negative everywhere, change f to $-f$).

3. Characterization of the effective normal velocity

We now use the results described above to find useful characterizations of the homogenized Hamiltonian in the problem motivated by phase boundary propagation where

$$H(x,p) = f(x)|p|.$$

We set

$$\bar{H}(p) = \bar{v}_n(p)|p| \tag{3.1}$$

and call $\bar{v}_n$ the effective normal velocity.

LEMMA 3.1. *If f is Lipschitz continuous, Y-periodic and $f > 0$, the effective Hamiltonian is given by*

$$\bar{H}(p) = \sup_\lambda \left(\cos(\lambda, p) \limsup_{D\to\infty} \frac{D}{\inf_{\gamma \in \mathcal{D}_D^{0,\lambda}} T(\gamma)} \right) |p|, \tag{3.2}$$

where

$$\cos(a,b) = \frac{a\cdot b}{|a||b|}$$

denotes the cosine of the angle formed by the vectors a and b, $\mathcal{D}_D^{x,\lambda}$ is the set of all H^1 paths connecting x with $x + (\lambda/|\lambda|)D$ and

$$T(\gamma) = \int_\gamma \frac{\mathrm{d}l}{f}$$

($\mathrm{d}l$ is the arclength).

Furthermore, $\bar{v}_n$ is homogeneous of degree zero, i.e. it is even and depends only on the direction $p/|p|$.

Proof. If v is a viscosity solution corresponding to p in (2.1), then, for any $\lambda > 0$, notice that λv is a solution corresponding to λp and $\bar{H}(\lambda p) = \lambda \bar{H}(p)$,

$$H(y, \lambda p + \lambda \nabla_y v) = f(y)|\lambda p + \lambda \nabla_y v| = \lambda f(y)|p + \nabla_y v| = \lambda H(y, p + \nabla_y v) = \lambda \bar{H}(p).$$

By the uniqueness of the cell problem, it follows that $\bar{v}_n$ is a function of direction only,

$$\bar{v}_n(p) = \bar{v}_n\left(\frac{p}{|p|}\right).$$

Moreover, proposition 2 in [12] ensures that $\bar{v}_n$ is even.

Using this property of $\bar{H}$, we get a first formula for its Legendre dual,

$$\bar{L}(\lambda) = \sup_p(\lambda \cdot p - \bar{H}(p)) = \sup_p |p|(|\lambda| \cos(\lambda, p) - \bar{v}_n(p)) = \begin{cases} 0 & \text{if } |\lambda| \leqslant F(\lambda), \\ \infty & \text{if } |\lambda| > F(\lambda), \end{cases} \tag{3.3}$$

where F is defined by

$$F(\lambda) = \inf_p \left\{ \frac{\bar{v}_n(p)}{\cos(\lambda, p)} : \cos(\lambda, p) > 0 \right\}.$$

Furthermore, since $\bar{H}$ is continuous and convex, it coincides with the dual of its dual, so

$$\bar{H}(p) = \sup_\lambda(\lambda \cdot p - \bar{L}(\lambda)) = \sup_{|\lambda| \leqslant F(\lambda)} |\lambda| \cos(\lambda, p)|p|.$$

Comparing this with (3.1), we infer

$$\bar{v}_n(p) = \sup_{|\lambda| \leqslant F(\lambda)} |\lambda| \cos(\lambda, p) = \sup_\lambda (F(\lambda) \cos(\lambda, p)). \tag{3.4}$$

An alternative formula for $\bar{L}$ may be obtained from E's relation (2.5). Using (2.4),

$$\bar{L}(\lambda) = \begin{cases} 0 & \text{if } \exists D_n \to \infty,\ \exists \phi_n \in H_0^1(0, D_n) \ni: |\lambda + \dot{\phi}_n| \leqslant f(\lambda t + \phi_n), \\ \infty & \text{otherwise.} \end{cases} \tag{3.5}$$

Now consider some λ with $|\lambda| > F(\lambda)$. By (3.3), $\bar{L}(\lambda) = \infty$, so (3.5) implies that, for any D large enough and for any ϕ in $H_0^1(0, D)$,

$$\exists (a, b) \subset (0, D) \ni: |\lambda + \dot{\phi}(t)| > f(\lambda t + \phi(t)) \quad \text{a.e. in } (a, b). \tag{3.6}$$

Having D and an arbitrary ϕ fixed, we pose the following initial-value problem,

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{f(\lambda t + \phi(t))}{|\lambda + \dot{\phi}(t)|}, \qquad t(0) = 0$$

on the largest interval on which $|\lambda + \dot{\phi}(t)|$ stays strictly positive and $t \leqslant D$. Since f is periodic and continuous and ϕ is in $H_0^1(0, D)$, the derivative of t is bounded from below by a strictly positive number. This ensures that t will eventually get to assume the value D. Set S to be the corresponding value of s.

We now show that $S > D$. Assume, by contradiction, that $S \leqslant D$. Then we can define a function in $H_0^1(0, D)$ by

$$\psi(s) = \begin{cases} \phi(t(s)) + \lambda(t(s) - s) & \text{if } s \leqslant S, \\ \lambda(\min_x f(x) - 1)s & \text{if } S < s \leqslant D, \end{cases}$$

which satisfies

$$\left|\lambda + \frac{\mathrm{d}\psi}{\mathrm{d}s}\right| \leqslant f(\lambda s + \psi(t(s))) \quad \text{a.e. in } [0, D]. \tag{3.7}$$

Since (3.6) and (3.7) contradict each other, it must be that $S > D$, which yields

$$D < \int_0^S \mathrm{d}s = \int_0^D \frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t = \int_0^D \frac{|\lambda + \dot{\phi}(t)|}{f(\lambda t + \phi(t))}\,\mathrm{d}t.$$

What we have proved up to now is the following: for any λ with $|\lambda| > F(\lambda)$, any sufficiently large D and any ϕ in $H_0^1(0, D)$, we have

$$|\lambda| > |\lambda| D \left(\int_0^D \frac{|\lambda + \dot{\phi}(t)|}{f(\lambda t + \phi(t))}\,\mathrm{d}t \right)^{-1}. \tag{3.8}$$

Equivalently,

$$F(\lambda) \geqslant \limsup_{D\to\infty} \sup_{\phi \in H_0^1(0,D)} |\lambda| D \left(\int_0^D \frac{|\lambda + \dot{\phi}(t)|}{f(\lambda t + \phi(t))}\,\mathrm{d}t \right)^{-1}. \tag{3.9}$$

We now aim for the converse inequality. Attempting the same trail, consider some λ with $|\lambda| < F(\lambda)$. By (3.3), $\bar{L}(\lambda) = 0$, so (3.5) implies that there exists a sequence $D_n \to \infty$ and $\phi_n \in H_0^1(0, D_n)$ such that $|\lambda + \dot{\phi}_n(t)| \leqslant f(\lambda t + \phi_n(t))$ a.e. in $[0, D_n]$. This yields

$$\int_0^{D_n} \frac{|\lambda + \dot{\phi}_n(t)|}{f(\lambda t + \phi_n(t))}\,\mathrm{d}t \leqslant D_n,$$

and hence

$$\sup_{\phi \in H_0^1(0,D_n)} |\lambda| D_n \left(\int_0^{D_n} \frac{|\lambda + \dot{\phi}_n(t)|}{f(\lambda t + \phi_n(t))}\,\mathrm{d}t \right)^{-1} \geqslant |\lambda|.$$

Thus

$$|\lambda| \leqslant \limsup_{D\to\infty} \sup_{\phi \in H_0^1(0,D)} |\lambda| D \left(\int_0^D \frac{|\lambda + \dot{\phi}(t)|}{f(\lambda t + \phi(t))}\,\mathrm{d}t \right)^{-1} \quad \forall \lambda \text{ with } |\lambda| < F(\lambda),$$

which gives the converse inequality of (3.9). We conclude that

$$F(\lambda) = \limsup_{D\to\infty} \sup_{\phi\in H_0^1(0,D)} |\lambda| D \left(\int_0^D \frac{|\lambda + \dot{\phi}(t)|}{f(\lambda t + \phi(t))} \,\mathrm{d}t \right)^{-1}.$$

Since, for any $\phi \in H_0^1(0,D)$, the function $\lambda t + \phi(t)$, $t \in (0,D)$, describes a H^1 path connecting 0 with $0 + (\lambda/|\lambda|)D$. We can rewrite this as

$$F(\lambda) = \limsup_{D\to\infty} \left(\frac{1}{D} \inf_{\gamma\in\mathcal{D}_D^{0,\lambda}} \int_\gamma \frac{\mathrm{d}l}{f} \right)^{-1}, \tag{3.10}$$

and using (3.4) we reach (3.2). □

This formula for $\bar{v}_n$ has an interesting interpretation. According to (3.10), $F(\lambda)$ is the fastest average velocity with which we can travel a long distance in the λ direction. Equation (3.2) then picks the direction λ such that the projected velocity in the direction p is the fastest. Therefore, $\bar{v}_n$ is the fastest projected average velocity in the direction p. In particular, a front need not directly develop with normal p. If a quicker neighbouring direction exists, the front will rather choose to develop portions with the corresponding normal at the microscopical level to obtain a faster propagation.

Lemma 3.2. *The function*

$$G(x,y) = \inf_{\phi\in H_0^1(0,1)} \int_0^1 \frac{|y - x + \dot{\phi}(t)|}{f(x + t(y-x) + \phi(t))} \,\mathrm{d}t$$

satisfies the Lipschitz continuity condition

$$|G(x_1,y_1) - G(x_2,y_2)| \leqslant \frac{|x_1 - x_2| + |y_1 - y_2|}{\inf_x f(x)}.$$

Proof. Given $\phi_1 \in H_0^1(0,1)$, define $\phi_2 \in H_0^1(0,1)$ by

$$x_2 + t(y_2 - x_2) + \phi_2(t) = \begin{cases} x_2 + 3t(x_1 - x_2) & \text{if } t \in [0, \frac{1}{3}], \\ x_1 + (3t-1)(y_1 - x_1) + \phi_1(3t-1) & \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ y_1 + (3t-2)(y_2 - y_1) & \text{if } t \in [\frac{2}{3}, 1]. \end{cases}$$

We have that

$$\begin{aligned} G(x_2,y_2) &\leqslant \int_0^1 \frac{|y_2 - x_2 + \dot{\phi}_2(t)|}{f(x_2 + t(y_2 - x_2) + \phi_2(t))} \,\mathrm{d}t \\ &= \int_0^{1/3} \frac{3|x_1 - x_2|}{f(x_2 + 3t(x_1 - x_2))} \,\mathrm{d}t \\ &\quad + \int_{1/3}^{2/3} \frac{3|y_1 - x_1 + \dot{\phi}_1(3t-1)|}{f(x_1 + (3t-1)(y_1 - x_1) + \phi_1(3t-1))} \,\mathrm{d}t \\ &\quad + \int_{2/3}^1 \frac{3|y_2 - y_1|}{f(y_1 + (3t-2)(y_2 - y_1))} \,\mathrm{d}t \end{aligned}$$

$$\leqslant \frac{|x_1 - x_2|}{\min_x f} + \int_0^1 \frac{|y_1 - x_1 + \dot{\phi}_1(t)|}{f(x_1 + t(y_1 - x_1) + \phi_1(t))}\,\mathrm{d}t + \frac{|y_1 - y_2|}{\min_x f}.$$

Since this inequality is satisfied for any $\phi_1 \in H_0^1(0,1)$, we have verified that

$$G(x_2, y_2) - G(x_1, y_1) \leqslant \frac{|x_1 - x_2|}{\min_x f} + \frac{|y_1 - y_2|}{\min_x f}.$$

By symmetry of x, y, we reach the conclusion of the lemma. □

LEMMA 3.3.

(i) *The starting point of the curves in the family* $\mathcal{D}_D^{0,\lambda}$ *in (3.2) is arbitrary,*

$$F(\lambda) = \limsup_{D\to\infty} \frac{D}{\inf_{\gamma\in\mathcal{D}_D^{0,\lambda}} T(\gamma)} = \limsup_{D\to\infty} \frac{D}{\inf_{\gamma\in\mathcal{D}_D^{\lambda}} T(\gamma)},$$

where $\mathcal{D}_D^{\lambda}$ *is the set of all* H^1 *paths connecting* x *with* $x + (\lambda/|\lambda|)D$ *for some* $x \in \mathbb{R}^N$.

(ii) *The* lim sup *in (3.2) is actually a limit.*

Proof. (i) By considering the $[0,1]$ parametrizations of the paths in $\mathcal{D}_D^{0,\lambda}$, we get

$$\inf_{\gamma\in\mathcal{D}_D^{0,\lambda}} T(\gamma) = \inf_{\phi\in H_0^1(0,1)} \int_0^1 \frac{|(\lambda/|\lambda|)D + \dot{\phi}(t)|}{f((\lambda/|\lambda|)D + \phi(t))}\,\mathrm{d}t = G\biggl(0, \frac{\lambda}{|\lambda|}D\biggr).$$

Then lemma 3.2 yields

$$\frac{1}{D} \inf_{\gamma\in\mathcal{D}_D^{x,\lambda}} T(\gamma) \leqslant \frac{1}{D} \inf_{\gamma\in\mathcal{D}_D^{0,\lambda}} T(\gamma) + \frac{2|x|}{D \min_y f(y)} \quad \forall D > 0 \quad \forall x \in \mathbb{R}^N. \tag{3.11}$$

By the definition of lim sup, there exists a sequence $D_n \to \infty$ such that

$$\limsup_{D\to\infty} \frac{D}{\inf_{\gamma\in\mathcal{D}_D^{0,\lambda}} T(\gamma)} = \lim_{n\to\infty} \frac{D_n}{\inf_{\gamma\in\mathcal{D}_{D_n}^{0,\lambda}} T(\gamma)}.$$

Since

$$\frac{2|x|}{D \min_y f(y)} \to 0 \quad \text{as } D \to \infty,$$

equation (3.11) implies

$$\begin{aligned}
\limsup_{D\to\infty} \frac{D}{\inf_{\gamma\in\mathcal{D}_D^{0,\lambda}} T(\gamma)} &\leqslant \lim_{n\to\infty} \frac{D_n}{\inf_{\gamma\in\mathcal{D}_{D_n}^{x,\lambda}} T(\gamma)} \\
&\leqslant \limsup_{D\to\infty} \frac{D}{\inf_{\gamma\in\mathcal{D}_D^{x,\lambda}} T(\gamma)} \quad \forall x \in \mathbb{R}^N.
\end{aligned}$$

The converse inequality follows in a similar manner, so we have proved

$$\limsup_{D\to\infty} \frac{D}{\inf_{\gamma\in\mathcal{D}_D^{0,\lambda}} T(\gamma)} = \limsup_{D\to\infty} \frac{D}{\inf_{\gamma\in\mathcal{D}_D^{x,\lambda}} T(\gamma)} \quad \forall x \in \mathbb{R}^N.$$

from which we reach our assertion.

(ii) Let $\xi = \lambda/|\lambda|$ and

$$L = \liminf_{D\to\infty} \frac{1}{D} \inf_{\gamma\in\mathcal{D}_D^{0,\lambda}} T(\gamma) = \liminf_{D\to\infty} \frac{G(0, D\xi)}{D}.$$

Fix some $\varepsilon > 0$. By the definition of lim inf, there exists some

$$d_\varepsilon \geqslant \frac{3\sqrt{N}}{\varepsilon \min_x f(x)}$$

for which

$$\left| L - \frac{G(0, d_\varepsilon\xi)}{d_\varepsilon} \right| \leqslant \tfrac{1}{3}\varepsilon.$$

Pick some integer k with

$$k \geqslant \frac{3}{\varepsilon \min_x f(x)}.$$

Let $D_\varepsilon = kd_\varepsilon$ and choose some arbitrary D with $D > D_\varepsilon$.

Now pick an integer m, with $m \geqslant k$, for which $D \in (md_\varepsilon, (m + 1)d_\varepsilon]$. Then lemma 3.2 implies

$$G(0, D\xi) \leqslant G(0, md_\varepsilon\xi) + \frac{D - md_\varepsilon}{\min_x f(x)} \leqslant G(0, md_\varepsilon\xi) + \frac{md_\varepsilon}{\min_x f(x)}.$$

Since $m \geqslant k$ and $k \geqslant 3/(\varepsilon \min_x f(x))$, this yields

$$\begin{aligned} \frac{G(0, D\xi)}{D} &\leqslant \frac{G(0, D\xi)}{md_\varepsilon} \\ &\leqslant \frac{G(0, md_\varepsilon\xi)}{md_\varepsilon} + \frac{1}{m \min_x f(x)} \\ &\leqslant \frac{G(0, md_\varepsilon\xi)}{md_\varepsilon} + \frac{1}{k \min_x f(x)} \\ &\leqslant \frac{G(0, md_\varepsilon\xi)}{md_\varepsilon} + \tfrac{1}{3}\varepsilon. \end{aligned} \tag{3.12}$$

On the other hand, we clearly have

$$G(0, md_\varepsilon\xi) \leqslant G(0, (m - 1)d_\varepsilon\xi) + G((m - 1)d_\varepsilon\xi, md_\varepsilon\xi). \tag{3.13}$$

Since $(m-1)d_\varepsilon \in \mathbb{R}^N$, there exists some $y \in \mathbb{Z}^N$ such that $|y-(m-1)d_\varepsilon| \leqslant \frac{1}{2}\sqrt{N}$. Due to the periodicity of f, we have that

$$G(y, y + d_\varepsilon\xi) = G(0, d_\varepsilon\xi).$$

By lemma 3.2 and this last equality, equation (3.13) becomes

$$\begin{aligned} G(0, md_\varepsilon\xi) &\leqslant G(0, (m - 1)d_\varepsilon\xi) + G(y, y + d_\varepsilon\xi) + 2\frac{|y - (m - 1)d_\varepsilon|}{\min_x f(x)} \\ &\leqslant G(0, (m - 1)d_\varepsilon\xi) + G(0, d_\varepsilon\xi) + \frac{\sqrt{N}}{\min_x f(x)}. \end{aligned}$$

Iterating this inequality, we obtain

$$G(0, md_\varepsilon\xi) \leqslant mG(0, d_\varepsilon\xi) + (m-1)\frac{\sqrt{N}}{\min_x f(x)}.$$

Recalling the two assumptions we made on d_ε,

$$\begin{aligned}\frac{G(0, md_\varepsilon\xi)}{md_\varepsilon} &\leqslant \frac{G(0, d_\varepsilon\xi)}{d_\varepsilon} + \frac{m-1}{m}\frac{\sqrt{N}}{d_\varepsilon \min_x f(x)} \\ &\leqslant L + \tfrac{1}{3}\varepsilon + \frac{\sqrt{N}}{d_\varepsilon \min_x f(x)} \\ &\leqslant L + \tfrac{2}{3}\varepsilon. \end{aligned}\tag{3.14}$$

Adding the inequalities (3.12) and (3.14), we get

$$\frac{G(0, D\xi)}{D} \leqslant L + \varepsilon = \liminf_{D\to\infty} \frac{G(0, D\xi)}{D} + \varepsilon \quad \forall T > T_\varepsilon,$$

which finally allows us to conclude that

$$\liminf_{D\to\infty} \frac{G(0, D\xi)}{D} = L = \lim_{D\to\infty} \frac{G(0, D\xi)}{D}.$$

This equality is sufficient to establish our assertion. □

We now present our main results.

THEOREM 3.4. *If f is Lipschitz continuous, Y-periodic and $f > 0$, the effective Hamiltonian is given by*

$$\bar{H}(p) = \sup_\lambda\left(|p|\cos(\lambda, p) \lim_{D\to\infty} \frac{D}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)}\right) = \lim_{D\to\infty} \sup_{\gamma\in\mathcal{D}_D} \frac{p\cdot d(\gamma)}{T(\gamma)}, \tag{3.15}$$

where $d(\gamma)$ is the vector joining the two endpoints of γ and $\mathcal{D}_D$ is the set of all H^1 paths with $|d(\gamma)| = D$.

The effective Hamiltonian is an even function of p and varies linearly with its length $|p|$.

Proof. The first equality is a direct consequence of the lemmas above. For the second, we need to prove that the limit can be interchanged with the supremum.

Since the expression inside the supremum depends only on the direction of λ, it is enough to consider only the unit vectors ($\lambda \in \partial B_1(0)$). Consider the inequality

$$\sup_{\lambda\in\partial B_1(O)} \frac{Dp\cdot\lambda}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)} \geqslant \frac{Dp\cdot\lambda}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)}.$$

It follows that

$$\liminf_{D\to\infty} \sup_{\lambda\in\partial B_1(O)} \frac{Dp\cdot\lambda}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)} \geqslant \lim_{D\to\infty} \frac{Dp\cdot\lambda}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)}.$$

since the right-hand side has a limit by lemma 3.3. This holds for any λ, so it follows that

$$\liminf_{D\to\infty} \sup_{\lambda\in\partial B_1(O)} \frac{Dp\cdot\lambda}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)} \geqslant \sup_{\lambda\in\partial B_1(0)} \lim_{D\to\infty} \frac{Dp\cdot\lambda}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)}. \tag{3.16}$$

We now prove the converse inequality. Consider the expression on the left-hand side of (3.16). By the definition of lim sup, we can find a sequence $D_n \to \infty$ such that

$$\liminf_{D\to\infty} \sup_{\lambda\in\partial B_1(O)} \frac{Dp\cdot\lambda}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)} = \lim_{n\to\infty} \sup_{\lambda\in\partial B_1(O)} \frac{D_n p\cdot\lambda}{\inf_{\gamma\in\mathcal{D}_{D_n}^\lambda} T(\gamma)}.$$

Since $\partial B_1(0)$ is compact and the expression inside the supremum is continuous with respect to λ by lemma 3.2, the supremum is attained at $\lambda_n \in B_1(0)$. We thus have two sequences, $D_n \to \infty$ and $\{\lambda_n\}_{n\in\mathbb{N}} \in \partial B_1(0)$, such that

$$\limsup_{D\to\infty} \sup_{\lambda\in\partial B_1(0)} \frac{D\lambda\cdot p}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)} = \lim_{n\to\infty} \frac{D_n\lambda_n\cdot}{\inf_{\gamma\in\mathcal{D}_{D_n}^{\lambda_n}} T(\gamma)}.$$

Using again the compactness of $\partial B_1(0)$, and restricting to a subsequence if necessary, the sequence λ_n converges to some $\lambda \in \partial B_1(0)$.

A direct application of lemma 3.2 gives

$$\begin{aligned} \inf_{\gamma\in\mathcal{D}_{D_n}^{\lambda_n}} T(\gamma) &= G(0, D_n\lambda_n) \\ &\geqslant G(0, D_n\lambda) - \frac{D_n|\lambda-\lambda_n|}{\min_x f(x)} \\ &= \inf_{\gamma\in\mathcal{D}_{D_n}^{\lambda}} T(\gamma) - \frac{D_n|\lambda-\lambda_n|}{\min_x f(x)}. \end{aligned}$$

Thus

$$\frac{\inf_{\gamma\in\mathcal{D}_{D_n}^{\lambda_n}} T(\gamma)}{D_n} \geqslant \frac{\inf_{\gamma\in\mathcal{D}_{D_n}^{\lambda}} T(\gamma)}{D_n} - \frac{|\lambda-\lambda_n|}{\min_x f(x)}$$

and, since $\lambda_n\cdot p \to \lambda\cdot p$ as $n\to\infty$,

$$\begin{aligned} \limsup_{D\to\infty} \sup_{\lambda\in\partial B_1(0)} \frac{D\lambda\cdot p}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)} &= \lim_{n\to\infty} \frac{D_n\lambda_n\cdot p}{\inf_{\gamma\in\mathcal{D}_{D_n}^{\lambda_n}} T(\gamma)} \\ &\leqslant \lim_{n\to\infty} \frac{D_n\lambda\cdot p}{\inf_{\gamma\in\mathcal{D}_{D_n}^{\lambda}} T(\gamma)} \\ &= \lim_{D\to\infty} \frac{D\lambda\cdot p}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)} \\ &\leqslant \sup_{\lambda\in\partial B_1(0)} \lim_{D\to\infty} \frac{D\lambda\cdot p}{\inf_{\gamma\in\mathcal{D}_D^\lambda} T(\gamma)}. \end{aligned}$$

This inequality and (3.16) conclude the proof. □

REMARK 3.5. The effective velocity behaves monotonically with respect to f.

Proof. Using (3.15), we see that $f_1 \leqslant f_2$ implies

$$\int_\gamma \frac{\mathrm{d}l}{f_1} \geqslant \int_\gamma \frac{\mathrm{d}l}{f_2} \quad \forall \gamma,$$

which yields

$$\sup_{\gamma \in \mathcal{D}_D} \frac{p \cdot d(\gamma)}{\int_\gamma (1/f_1)\,\mathrm{d}l} \leqslant \sup_{\gamma \in \mathcal{D}_D} \frac{p \cdot d(\gamma)}{\int_\gamma (1/f_2)\,\mathrm{d}l} \quad \forall D.$$

Hence $H_1(p) \leqslant H_2(p)\ \forall p \in \mathbb{R}^N$. ◻

THEOREM 3.6. *If f is Lipschitz continuous, Y-periodic and $f > 0$, the effective Hamiltonian is given by*

$$\bar{H}(p) = \lim_{T \to \infty} \sup_{\gamma \in \mathcal{T}_T} \frac{p \cdot d(\gamma)}{T(\gamma)}, \tag{3.17}$$

where $\mathcal{T}_T$ is the set of all H^1 paths with $T(\gamma) = T$.

Proof. Define

$$S(T) = \sup_{\gamma \in \mathcal{T}_T} \frac{p \cdot d(\gamma)}{T(\gamma)}.$$

Fix some positive small ε. For any T, we may find some $\gamma_\varepsilon \in \mathcal{T}_T$ such that

$$S(T) - \tfrac{1}{2}\varepsilon < \frac{p \cdot d(\gamma_\varepsilon)}{T(\gamma_\varepsilon)} \leqslant \sup_{\gamma \in \mathcal{D}_{|d(\gamma_\varepsilon)|}} \frac{p \cdot d(\gamma)}{T(\gamma)}. \tag{3.18}$$

Consider the path $\tilde{\gamma}$ described by $y(s)p$ for $s \in [0, T]$, where y is the solution of the initial-value problem

$$\dot{y}(s) = \frac{f(y(s)p)}{|p|}, \qquad y(0) = 0.$$

It is easy to see that $\tilde{\gamma} \in \mathcal{T}_T$ and that

$$|d(\gamma)| = \int_\gamma \mathrm{d}l = \int_0^T |\dot{y}(s)p|\,\mathrm{d}s \geqslant T \min_x f(x).$$

We infer that $S(T)$ is bounded from below for any T,

$$S(T) \geqslant |p| \min_x f(x).$$

Plugging this into (3.18), we have found some $\gamma_\varepsilon \in \mathcal{T}_T$ such that

$$|d(\gamma_\varepsilon)| > \frac{T(|p| \min_x f(x) - \frac{1}{2}\varepsilon)}{|p|}. \tag{3.19}$$

Now, using (3.15), we find a positive D_ε such that

$$\sup_{\gamma \in \mathcal{D}_D} \frac{p \cdot d(\gamma)}{T(\gamma)} < \bar{H}(p) + \tfrac{1}{2}\varepsilon \quad \forall D \geqslant D_\varepsilon. \tag{3.20}$$

Let

$$T_\varepsilon = \frac{|p| D_\varepsilon}{|p| \min_x f(x) - \frac{1}{2}\varepsilon}.$$

For any $T \geqslant T_\varepsilon$, it follows from (3.19) that

$$|d(\gamma_\varepsilon)| > \frac{T(|p| \min_x f(x) - \frac{1}{2}\varepsilon)}{|p|} \geqslant \frac{T_\varepsilon(|p| \min_x f(x) - \frac{1}{2}\varepsilon)}{|p|} = D_\varepsilon,$$

so (3.20) and (3.18) yield

$$S(T) < \bar{H}(p) + \varepsilon \quad \forall T \geqslant T_\varepsilon.$$

Thus

$$\limsup_{T\to\infty} S(T) \leqslant \bar{H}(p). \tag{3.21}$$

We now consider the converse inequality. Fix some small positive ε and use (3.15) to obtain D_ε for which

$$\bar{H}(p) - \varepsilon < \sup_{\gamma \in \mathcal{D}_{D_\varepsilon}} \frac{p \cdot d(\gamma)}{T(\gamma)}.$$

Then there exists some path $\gamma_\varepsilon \in \mathcal{D}_{D_\varepsilon}$ such that

$$\bar{H}(p) - \varepsilon < \frac{p \cdot d(\gamma_\varepsilon)}{T(\gamma_\varepsilon)} \leqslant S(T_\varepsilon),$$

where $T_\varepsilon = T(\gamma_\varepsilon)$. But

$$T(\gamma_\varepsilon) \geqslant \frac{\int_{\gamma_\varepsilon} \mathrm{d}l}{\max_x f(x)} \geqslant \frac{|d(\gamma_\varepsilon)|}{\max_x f(x)} = \frac{D_\varepsilon}{\max_x f(x)},$$

so we have found some T_ε larger than $D_\varepsilon/(\max_x f(x))$ for which

$$\bar{H}(p) - \varepsilon < S(T_\varepsilon).$$

This is enough to infer that

$$\limsup_{T\to\infty} S(T) \geqslant \bar{H}(p).$$

All that remains is to show that the lim sup above is a limit, and this can be done as in the proof of lemma 3.3 (ii). □

Corollary 3.7. *An alternative representation for the effective Hamiltonian is*

$$\bar{H}(p) = \lim_{T\to\infty} \sup\left(\frac{p \cdot (x(T) - x(0))}{T} \;\middle|\; x \in H^1(0,T),\ |\dot{x}| = f(x) \right). \tag{3.22}$$

Proof. For any path in $\mathcal{T}_T$, we choose the parametrization $x(s)$, $s \in [0, S]$, given by

$$\frac{\mathrm{d}s}{\mathrm{d}l} = \frac{1}{f(x(s))}.$$

Such parametrizations always lie on the interval $[0, T]$, since

$$S = \int_0^S \mathrm{d}s = \int_\gamma \frac{\mathrm{d}l}{f} = T(\gamma) = T,$$

and they satisfy

$$|\dot{x}(s)| = \frac{\mathrm{d}l}{\mathrm{d}s} = f(x(s)).$$

Conversely, any $x \in H^1(0, T)$ that satisfies $|\dot{x}| = f(x)$ describes a path γ with

$$T(\gamma) = \int_\gamma \frac{\mathrm{d}l}{f} = \int_0^T \frac{|\dot{x}(s)|}{f(x(s))}\,\mathrm{d}s = T,$$

so (3.22) follows from theorem 3.6. □

This corollary allows us to extend a variational principle for the effective Hamiltonian that was originally obtained by Concordel in [5] for Hamiltonians with superlinear growth as $|p| \to \infty$ to our case where the Hamiltonian grows linearly. The result is false for sublinear growth (a simple counterexample is given by $H(x, p) = |p|^{1/2}$).

THEOREM 3.8. *If f is Lipschitz continuous, Y-periodic and $f > 0$, the effective Hamiltonian is given by*

$$\bar{H}(p) = \lim_{T\to\infty} \sup_{x\in H^1(0,T)} \frac{1}{T} \int_0^T [p \cdot \dot{x}(t) - L(\dot{x}(t), x(t))]\,\mathrm{d}t, \tag{3.23}$$

where L is the Lagrangian described in (2.4).

Proof. Denote by $S(p, T)$ the supremum occurring in the right-hand term of (3.23). Using (2.4),

$$S(p, T) = \sup\left\{\frac{1}{T}\int_0^T p \cdot \dot{x}(t)\,\mathrm{d}t \;\middle|\; x \in H^1(0, T),\ |\dot{x}| \leqslant f(x)\right\}.$$

Pick some $x \in H^1(0, T)$ with $|\dot{x}| \leqslant f(x)$ on $[0, T]$ and pose the following initial-value problem on $[0, T]$:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{|\dot{x}(t)|}{f(x(t))}, \qquad s(0) = 0.$$

Since $|\dot{x}| \leqslant f(x)$ on $[0, T]$ and $s(0) = 0$,

$$T = \int_0^T \mathrm{d}t \geqslant \int_0^T \frac{|\dot{x}(t)|}{f(x(t))}\,\mathrm{d}t = \int_0^{s(T)} \mathrm{d}s = s(T) = S,$$

and we have equality if and only if $|\dot{x}| = f(x)$ on $[0, T]$. Moreover, due to the strict positivity of f, the function $s(t)$ is invertible, so we may define $y \in H^1(0, T)$ by

$$y(s) = \begin{cases} x(t(s)) & \text{if } s \in [0, S], \\ x(T) + (s - S)\min_z f(z)p & \text{if } s \in [S, T]. \end{cases}$$

Then

$$\begin{aligned}\int_0^T p\cdot\dot{y}(s)\,\mathrm{d}s &= \int_0^S p\cdot\dot{y}(s)\,\mathrm{d}s + \int_S^T p\cdot\dot{y}(s)\,\mathrm{d}s\\ &= \int_0^T p\cdot\dot{x}(t)\,\mathrm{d}t + (T-S)\min_z f(z)p^2\\ &\geqslant \int_0^T p\cdot\dot{x}(t)\,\mathrm{d}t.\end{aligned}$$

Furthermore, the inequality is strict unless $T=S$. All this leads to the conclusion that, for the evaluation of $S(p,T)$, it is sufficient to consider those functions $x\in H^1(0,T)$ with $|\dot{x}|=f(x)$ on $[0,T]$. Then

$$\begin{aligned}S(p,T) &= \sup\left\{\frac{1}{T}\int_0^T p\cdot\dot{x}(t)\,\mathrm{d}t \;\middle|\; x\in H^1(0,T),\ |\dot{x}|=f(x)\right\}\\ &= \sup\left\{\frac{p\cdot|x(T)-x(0)|}{T} \;\middle|\; x\in H^1(0,T),\ |\dot{x}|=f(x)\right\}.\end{aligned}$$

Hence (3.22) implies (3.23). □

THEOREM 3.9. *If f is Lipschitz continuous, Y-periodic and $f>0$, the effective Hamiltonian is given by*

$$\bar{H}(p) = \lim_{P\to\infty}\sup_{\gamma\in\mathcal{P}_P}\frac{p\cdot d(\gamma)}{T(\gamma)}, \tag{3.24}$$

where $\mathcal{P}_P$ is the set of all H^1 paths with $p\cdot d(\gamma)=P$.

Proof. Define

$$S(P) = \sup_{\gamma\in\mathcal{P}_P}\frac{p\cdot d(\gamma)}{T(\gamma)}.$$

Fix some positive small ε. Using (3.15), we find a positive D_ε such that

$$\sup_{\gamma\in\mathcal{D}_D}\frac{p\cdot d(\gamma)}{T(\gamma)} < \bar{H}(p)+\tfrac{1}{2}\varepsilon \quad \forall D\geqslant D_\varepsilon. \tag{3.25}$$

Let $P_\varepsilon = |p|D_\varepsilon$. For any $P\geqslant P_\varepsilon$, we may find some $\gamma_\varepsilon\in\mathcal{P}_P$ such that

$$S(P)-\tfrac{1}{2}\varepsilon < \frac{p\cdot d(\gamma_\varepsilon)}{T(\gamma_\varepsilon)} < \sup_{\gamma\in\mathcal{D}_{|d(\gamma_\varepsilon)|}}\frac{p\cdot d(\gamma)}{T(\gamma)}. \tag{3.26}$$

But

$$|d(\gamma_\varepsilon)| \geqslant \frac{p\cdot d(\gamma_\varepsilon)}{|p|} = \frac{P}{|p|} \geqslant \frac{P_\varepsilon}{|p|} = D_\varepsilon.$$

So (3.25) and (3.26) yield

$$S(P) < \bar{H}(p)+\varepsilon \quad \forall P\geqslant P_\varepsilon,$$

which implies

$$\limsup_{P\to\infty} S(P) \leqslant \bar{H}(p). \tag{3.27}$$

We now consider the converse inequality. Fix some small positive ε and use (3.15) to obtain D_ε for which

$$\bar{H}(p) - \varepsilon < \sup_{\gamma \in \mathcal{D}_{D_\varepsilon}} \frac{p \cdot d(\gamma)}{T(\gamma)}.$$

Then there exists some path $\gamma_\varepsilon \in \mathcal{D}_{D_\varepsilon}$ such that

$$\bar{H}(p) - \varepsilon < \frac{p \cdot d(\gamma_\varepsilon)}{T(\gamma_\varepsilon)} \leqslant S(P_\varepsilon), \tag{3.28}$$

where $P_\varepsilon = p \cdot d(\gamma_\varepsilon)$. But

$$T(\gamma_\varepsilon) \geqslant \frac{\int_{\gamma_\varepsilon} \mathrm{d}l}{\max_x f(x)} \geqslant \frac{|d(\gamma_\varepsilon)|}{\max_x f(x)} = \frac{D_\varepsilon}{\max_x f(x)},$$

so (3.28) gives

$$P_\varepsilon > (\bar{H}(p) - \varepsilon)\frac{D_\varepsilon}{\max_x f(x)}.$$

Thus we have found some P_ε larger than $(\bar{H}(p) - \varepsilon)(D_\varepsilon/(\max_x f(x)))$ for which

$$\bar{H}(p) - \varepsilon < S(P_\varepsilon).$$

This is enough to infer that

$$\limsup_{P \to \infty} S(P) \geqslant \bar{H}(p).$$

Finally, we can show that lim sup above is a limit as in the proof of lemma 3.3 (ii). □

THEOREM 3.10. *The effective normal velocity is continuous with respect to variations of f in $C(\mathbb{R}^N)$.*

Proof. Consider two Y-periodic continuous functions, f_1 and f_2, that are close in the $C(\mathbb{R}^N)$ topology,

$$\|f_1 - f_2\| = \sup_{x \in Y} |f_1(x) - f_2(x)| < \varepsilon$$

for some small positive ε, and set

$$\begin{aligned} m_i &= \min_x f_i(x), \\ M_i &= \max_x f_i(x), \\ m &= \min(m_1, m_2), \\ M &= \max(M_1, M_2), \\ H_i(p) &= \lim_{P \to \infty} \sup_{\gamma \in \mathcal{P}_P} \frac{P}{\int_\gamma (1/f_i)\,\mathrm{d}l}. \end{aligned}$$

For fixed $|p|$ and P, we obtain a bound for the supremum in (3.24) by considering the segment $[0, P(p/|p|)]$,

$$S(P) = \sup_{\gamma \in \mathcal{P}_P} \frac{P}{T(\gamma)} \geqslant |p| \min_x f(x).$$

Thus, for the evaluation of H_1 and H_2 above, we may restrict the classes $\mathcal{P}_P$ by imposing the supplementary inequality

$$\int_\gamma \mathrm{d}l \leqslant \frac{MP}{m|p|}.$$

Then, for any path $\gamma \in \mathcal{P}_P$, we have

$$\left|\frac{\int_\gamma (1/f_1)\,\mathrm{d}l}{P} - \frac{\int_\gamma (1/f_2)\,\mathrm{d}l}{P}\right| \leqslant \frac{\|1/f_1 - 1/f_2\| \int_\gamma \mathrm{d}l}{P} \leqslant \frac{M\varepsilon}{|p|m^3}.$$

This implies

$$\left|\inf_{\gamma\in\mathcal{P}_P} \frac{\int_\gamma (1/f_1)\,\mathrm{d}l}{P} - \inf_{\gamma\in\mathcal{P}_P} \frac{\int_\gamma (1/f_2)\,\mathrm{d}l}{P}\right| \leqslant \frac{M\varepsilon}{|p|m^3}.$$

Since the inequality above is uniform with respect to P, it remains true even in the limit, proving our continuity assertion. $\square$

Remark 3.11. Theorems 3.4, 3.6, 3.8, 3.9 and 3.10 and remark 3.5 can easily be extended to the anisotropic case. More exactly, if we allow f to have some directional dependence ($f(x,p) = f(x, p/|p|)$), the only modification in the formulae for the effective Hamiltonian (3.15), (3.17), (3.22) and (3.24) is that f should be replaced with

$$\tilde{f}(x,q) = \sup_{\cos(p,q)>0} \frac{f(x,p)}{\cos(p,q)}.$$

4. Some bounds

We now obtain some bounds that are sharper than (2.3). We derive these bounds for F, which is related to $\bar{v}_n$ and $\bar{H}$ through (3.4).

We start with a lower bound. Pick some λ with $|\lambda| > F(\lambda)$, and write (3.8) for $\phi = 0$ to get

$$D < |\lambda| \int_0^D \frac{1}{f(\lambda t)}\,\mathrm{d}t \leqslant |\lambda| \sup_{\phi\in H_0^1(0,D)} \int_0^D \frac{1}{f(\lambda t + \phi(t))}\,\mathrm{d}t.$$

Since this is true for any λ with $|\lambda| > F(\lambda)$, we conclude that we must have

$$F(\lambda) \geqslant D\biggl(\int_0^D \frac{1}{f(\lambda t)}\,\mathrm{d}t\biggr)^{-1} \geqslant D \inf_{\phi\in H_0^1(0,D)} \biggl(\int_0^D \frac{1}{f(\lambda t + \phi(t))}\,\mathrm{d}t\biggr)^{-1}.$$

This relation holds for any D sufficiently large, so it yields the following bound on F:

$$\begin{aligned} F(\lambda) &\geqslant \limsup_{D\to\infty} \biggl(D\biggl(\int_0^D \frac{1}{f(\lambda t)}\,\mathrm{d}t\biggr)^{-1}\biggr) \\ &\geqslant \limsup_{D\to\infty} \biggl(D \inf_{\phi\in H_0^1(0,D)} \biggl(\int_0^D \frac{1}{f(\lambda t + \phi(t))}\,\mathrm{d}t\biggr)^{-1}\biggr). \end{aligned} \tag{4.1}$$

For an upper bound, let λ be such that $|\lambda| < F(\lambda)$. By (3.3), $\bar{L}(\lambda) = 0$, so (3.5) implies that there exists a sequence $D_n \to \infty$ and $\phi_n \in H_0^1(0, D_n)$ such that $|\lambda + \dot{\phi}_n(t)| \leqslant f(\lambda t + \phi_n(t))$ a.e. in $[0, D_n]$. This yields

$$\begin{aligned}|\lambda| &= \left|\frac{1}{D_n}\int_0^{D_n}(\lambda + \dot{\phi}_n(t))\,\mathrm{d}t\right| \\ &\leqslant \frac{1}{D_n}\int_0^{D_n}|\lambda + \dot{\phi}_n(t)|\,\mathrm{d}t \\ &\leqslant \frac{1}{D_n}\int_0^{D_n} f(\lambda t + \phi_n(t))\,\mathrm{d}t \\ &\leqslant \frac{1}{D_n}\sup_{\phi\in H_0^1(0,D_n)}\int_0^{D_n} f(\lambda t + \phi(t))\,\mathrm{d}t.\end{aligned}$$

This relation stays true for a subsequence $D_n \to \infty$, so we must have that

$$|\lambda| \leqslant \limsup_{D\to\infty}\frac{1}{D}\sup_{\phi\in H_0^1(0,D)}\int_0^{D} f(\lambda t + \phi(t))\,\mathrm{d}t.$$

Since this holds for any λ with $|\lambda| < F(\lambda)$, we conclude that

$$F(\lambda) \leqslant \limsup_{D\to\infty}\frac{1}{D}\sup_{\phi\in H_0^1(0,D)}\int_0^{D} f(\lambda t + \phi(t))\,\mathrm{d}t. \tag{4.2}$$

5. Examples

We now discuss the issues that arise when f changes sign and also demonstrate the usefulness of our formulae through examples in $\mathbb{R}^2$. While deriving these formulae, we assumed that f was Lipschitz continuous and strictly positive. We note, however, that the formulae themselves remain well defined, even when we relax these assumptions. Therefore, in this section, we will use these formulae with all functions f for which they are well defined.

We begin by discussing three examples when f changes sign. Let

$$\mathcal{F}^+ = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{F}^- = \{x : f(x) < 0\}.$$

We have one of the following three interesting situations.

Example 5.1. We start by looking at the case when both $\mathcal{F}^+$ and $\mathcal{F}^-$ percolate. As a representative example, consider the normal velocity law

$$f(x) = a\cos(\tfrac{1}{2}\pi x_1), \quad x \in \mathbb{R}^2, \tag{5.1}$$

with given $a > 0$. In such a situation, we show that the existence of a moving homogenized front fails either due to the creation of very dense oscillations or due to trapping.

To see the former, let us start with an initial front $\{x_2 = 0\}$. The evolution of such a front is depicted in figure 2. The lines where v_n vanishes are shown as continuous lines and the evolving front is pictured at three different instants of

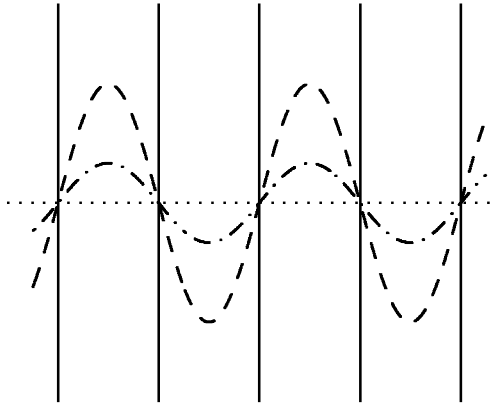

Figure 2. Propagation of a front with normal velocity given by (5.1).

time: the dotted line represents the initial position, followed chronologically by the dash–dot line and the dashed line. The front is pinned at all the points $x = (x_1, x_2)$ satisfying $x_1 \in \varepsilon(\mathbb{Z}\setminus 2\mathbb{Z})$ and $x_2 = 0$. On the other hand, the normal to the interface at the points x with $x_1 \in 2\varepsilon\mathbb{Z}$ is $\pm e_2$ by symmetry. Therefore, points on the front with $x_1 \in 4\varepsilon\mathbb{Z}$ moves in the e_2-direction, while those with $x_1 \in \varepsilon(2\mathbb{Z} \setminus 4\mathbb{Z})$ will move in the $-e_2$-direction with constant velocity a. It is now easy to see that, as ε decreases, the front assumes an oscillatory shape with constant amplitude and frequency of order ε^{-1}. Obviously, it cannot converge uniformly to any homogenized front. Instead, it converges to a set with an interior $\{|x_2| \leqslant at\}$, which is growing with velocity a.

To see trapping in this same example, consider the initial front $\{x_1 = 0\}$. The normal velocity is zero and does not move for all time. Furthermore, if we consider a translated initial front $\{x_1 = c\}$ for some constant c, we see that it remains trapped inside$\{|x_1 - c| < \varepsilon\}$ and converges to a stationary front $\{x_1 = c\}$.

Let us now examine this example with our formulae. First consider $p = e_2$, where we found that the front develops oscillations and converges to a set with an interior. Since $f(x) \leqslant a$ for all x, it follows from the elementary bound that

$$\bar{v}_n(p) \leqslant a \quad \forall p \in \mathbb{R}^2.$$

On the other hand, picking γ in (3.15) to be the vertical segment of length D parallel with e_2 and originating in 0, we get

$$\bar{H}(e_2) \geqslant a = a|e_2|.$$

Together they imply

$$\bar{v}_n(e_2) = a.$$

Note that this is the normal velocity with which the set with interior expands with time.

Now turn to $p = e_1$, where we found that the front gets trapped. Note that, for any $P > 2$, any path γ that belongs to $\mathcal{P}_P$ has to intersect one of the lines $x_1 \in (2\mathbb{Z} - 1)$, and hence it will contain a point where f vanishes. Thus the sup occurring in (3.9) is 0 for any $P > 2$, which implies that

$$\bar{v}_n(e_1) = 0.$$

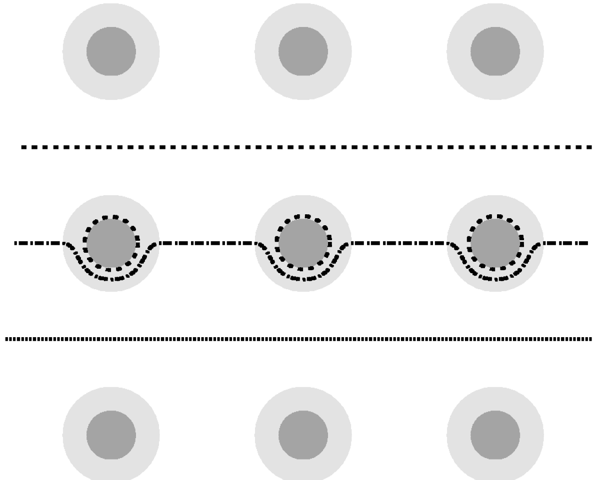

Figure 3. Propagation of a front with normal velocity given by (5.2).

As we have seen above, this is indeed the effective normal velocity in the e_1-direction, so our results have provided the correct answer.

Example 5.2. We now look at the case when neither $\mathcal{F}^+$ nor $\mathcal{F}^-$ percolate. An example of a normal velocity law that falls within this case is

$$f(x) = a\cos(\tfrac{1}{2}\pi x_1)\cos(\tfrac{1}{2}\pi x_2), \quad x \in \mathbb{R}^2.$$

We argue that the homogenized front is stationary. Indeed, since $\mathcal{F}^+$ does not percolate, all of its connected subsets must be bounded. Moreover, by periodicity of f, all the diameters of the connected subsets of $\mathcal{F}^+$ must have a common upper bound. By repeating this argument for $\mathcal{F}^-$, we obtain a positive real d_0 larger than the diameter of any connected subset on which f does not change sign. Any initial front $\mathcal{S}$ will then be confined to the region $\{d(x,\mathcal{S}) < \varepsilon d_0\}$. As ε decreases to 0, this region shrinks to $\mathcal{S}$ and traps the homogenized front.

We now examine this with our formula (3.9) to calculate $\bar{v}_n(p)$ for some $p \in \mathbb{R}^2$. Using the same line of reasoning as in the previous example, for any $P > 2\sqrt{2}$, any path γ that belongs to $\mathcal{P}_P$ will contain a point where f vanishes, so the sup occurring in (3.9) is 0 for any $P > 2\sqrt{2}$, which implies that

$$\bar{v}_n(p) = 0 \quad \forall p \in \mathbb{R}^2,$$

in agreement with our earlier discussion.

Example 5.3. Finally, let us consider the case when $\mathcal{F}^+$ percolates but $\mathcal{F}^-$ does not. An illustrative example for this case is

$$f(x) = \begin{cases} -1 + 8|x - (\frac{1}{2},\frac{1}{2})| & \text{if } x \in B_{1/4}(\frac{1}{2},\frac{1}{2}), \\ 1 & \text{if } x \in Y_2 \setminus B_{1/4}(\frac{1}{2},\frac{1}{2}), \end{cases} \tag{5.2}$$

extended by periodicity to all $\mathbb{R}^2$. This is shown in figure 3. The function f takes the value 1 outside the circles, and decreases linearly with radius to -1 at the centre.

Thus f is positive but decreasing in the light grey annuli, and negative in the dark grey circles.

Assume that we start with the initial front given by $\{x_2 = 0\}$, as shown in the figure with the dotted line. It translates with uniform velocity 1 in the e_2-direction until it touches the light grey annuli. Then regions of the front inside slow down, but those outside continue to propagate with normal velocity 1. This is indicated by the dash–dot line in the figure. In particular, we see from symmetry that all points with $x_1 \in \varepsilon\mathbb{Z}$ move in the e_2-direction with constant velocity 1 for all time. These points continue to drag the front forward, and as these points go past the circles, the front engulfs the dark grey circle, eventually splitting into multiple pieces: a leading front and traces around the dark grey circles, as shown by the dashed lines in the figure. The leading front continues to propagate past successive rows of circles while shedding these traces. Note that the velocity of the points with $x_1 \in \varepsilon\mathbb{Z}$ on this leading front is always 1 in the e_2-direction.

As ε decreases to 0, the number of such trace surfaces increases proportionally with ε^{-1}, and thus one can not use the level-set formulation directly to obtain a homogenized front. We wish to neglect the trace surfaces left behind, define the homogenized front by the leading part and estimate its effective velocity. One intuitive way of doing all this is to imitate Soravia's definition 4.2 in [13] and consider the limit, as ε decreases to 0, of the evolutions with speed $\max(f, 0) + \varepsilon$. It is possible to use Lions's theorem 1.12 in [11] and replace the negative parts of f with 0. The addition of the small positive constant ε deletes the trace surfaces left behind without considerably altering the velocity of the leading front, as can be seen in theorem 3.10.

We now examine this with our formulae and bounds. The inequality $f(x) \leqslant 1$ provides the bound $\bar{v}_n(p) \leqslant 1$ for all $p \in \mathbb{R}^2$. On the other hand, we see by choosing γ in (3.15) to be the vertical segment of length D parallel with e_2 and originating in 0 that $\bar{H}(e_2) \geqslant 1$. Therefore, we conclude that

$$\bar{v}_n(e_2) = 1.$$

We see that our formula is able to ignore the traces that the front leaves around the circles where f vanishes and pick up the velocity with which the leading part propagates.

We now turn to examples where f is strictly positive and demonstrate the usefulness of our formulae.

EXAMPLE 5.4. Consider a laminated composite with the normal velocity law

$$f(x) = \begin{cases} v_1 & \text{if } x \in [0, \mu] \times [0, 1], \\ v_2 & \text{if } x \in (\mu, 1) \times [0, 1], \end{cases}$$

extended by periodicity to all $\mathbb{R}^2$, where $v_1 > v_2 > 0$, $\mu \in (0, 1)$. We now compute $\bar{v}_n(p)$ using lemmas 3.1 and 3.3.

The first step is to compute $F(\lambda)$ for $\lambda \in \partial B_1(0)$ using lemma 3.3,

$$F(\lambda) = \lim_{D\to\infty} \frac{D}{\inf_{\gamma \in \mathcal{D}_D^{0,\lambda}} T(\gamma)}.$$

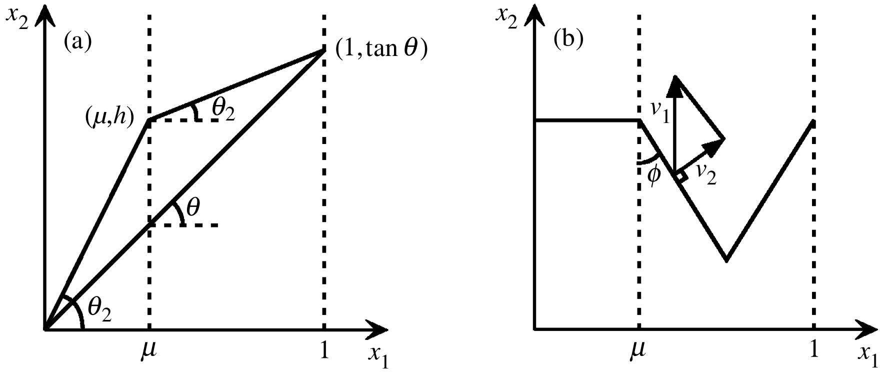

Figure 4. (a) Finding the optimal path for D_1 and (b) the front for $\lambda = e_2$.

Set $\lambda = (\sin\theta, \cos\theta)$. By symmetry, it is enough to restrict λ to the first quadrant, i.e. $\theta \in [0, \frac{1}{2}\pi]$. We further assume $\theta < \frac{1}{2}\pi$ for now, and deal with the case $\theta = \frac{1}{2}\pi$ separately. To evaluate the limit above, we consider the sequence $D_n = n/\cos\theta$, so that the path has to traverse exactly through n stripes in the e_1-direction. Now consider the problem of finding the optimal problem in the infimum above. If f were constant, then this problem reduces to that of finding the minimum distance resulting in optimal paths that are linear. Using this argument locally, it is easy to see that when f is piecewise constant as in the current example, then the optimal path is also piecewise constant.

Let us consider the problem for $D_1 = 1/\cos\theta$, where we have to find an optimal path joining $(0, 0)$ with $(1, \tan\theta)$, as shown in figure 4a. It is clear that the optimal path consists of two linear segments: one for $x_1 < \mu$ where $f = v_1$ and another for $x_1 > \mu$ where $f = v_2$. Therefore, the problem of finding the optimal path reduces to that of finding the height h at which this path intersects the vertical line $x_1 = \mu$. It is a matter of elementary geometry to see that

$$T(\gamma) = \frac{\sqrt{\mu^2 + h^2}}{v_1} + \frac{\sqrt{(1-\mu)^2 + (\tan\theta - h)^2}}{v_2},$$

so that the optimal h satisfies

$$\frac{h}{v_1\sqrt{\mu^2 + h^2}} = \frac{h - \tan\theta}{v_2\sqrt{(1-\mu)^2 + (\tan\theta - h)^2}}.$$

Introducing the angles θ_1, θ_2 that the segments make with the horizontal, we can rewrite this as

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2},$$

which is the familiar law of sines that we find in refraction.

For $n > 1$, the optimal path will have to obey this law at every jump point of f. But the only path that satisfies this is the one constructed by n repetitions of the

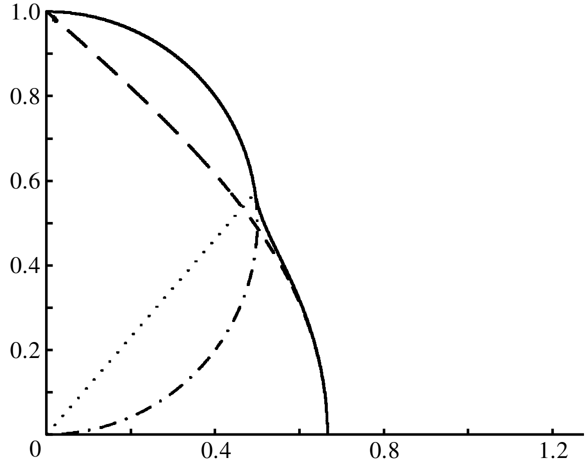

Figure 5. Polar plot of the effective normal velocity for a laminate with $\mu = 0.5$, $v_1 = 1$, $v_2 = 0.5$ is shown as the solid curve. The dashed curve shows the polar plot of F.

two-segment trajectory obtained for $n = 1$. Therefore, we conclude that

$$\begin{aligned}\inf_{\gamma\in\mathcal{D}^{0,\lambda}_{D_n}} T(\gamma) &= n\left(\frac{\sqrt{\mu^2+h^2}}{v_1} + \frac{\sqrt{(1-\mu)^2+(\tan\theta-h)^2}}{v_2}\right)\\ &= n\left(\frac{\mu}{v_1\cos\theta_1} + \frac{1-\mu}{v_2\cos\theta_2}\right).\end{aligned}$$

Substituting this back in the formula for F and passing to the limit $n\to\infty$, we see that

$$F(\lambda) = F(\cos\theta,\sin\theta) = \frac{1}{\cos\theta(\mu/(v_1\cos\theta_1) + (1-\mu)/(v_2\cos\theta_2))}, \tag{5.3}$$

where θ_1 and θ_2 are determined by

$$\mu\tan\theta_1 + (1-\mu)\tan\theta_2 = \tan\theta, \qquad \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}. \tag{5.4}$$

The system (5.4) can be solved numerically for given μ, v_1, v_2, and then F and $\bar{v}_n$ can be calculated from (5.3) and (3.4), respectively. Figure 5 shows the polar plot of F and $\bar{v}_n$ for $\mu = 0.5$, $v_1 = 1$, $v_2 = 0.5$ as the dashed and solid lines, respectively.

The remaining case $\theta = \frac{1}{2}\pi$ or $\lambda = e_2$ is simple. Appealing to lemma 3.3, we write

$$F(\lambda) = \lim_{D\to\infty}\frac{D}{\inf_{\gamma\in\mathcal{D}^{y,\lambda}_D} T(\gamma)},$$

with $y = (\frac{1}{2}\mu, 0)$. Recalling that the optimal paths must be linear in regions of constant f and that $v_1 > v_2$, we easily conclude that $F(e_2) = v_1$. Furthermore, for any $\theta < \frac{1}{2}\pi$, we can use (5.4) and $v_1 > v_2$ to see that

$$\begin{aligned}\frac{\tan\theta}{\mu/(v_1\cos\theta_1)+(1-\mu)/(v_2\cos\theta_2)} &\leqslant v_1\frac{\tan\theta}{\mu/\cos\theta_1+(1-\mu)/\cos\theta_2}\\ &\leqslant v_1\frac{\tan\theta}{\mu\sin\theta_1/\cos\theta_1+((1-\mu)\theta_2)/\cos\theta_2}\\ &= v_1,\end{aligned}$$

which may be rewritten as

$$F(\cos\theta,\sin\theta)\sin\theta \leqslant F(e_2).$$

We conclude that

$$\bar{v}_n(e_2) = v_1. \tag{5.5}$$

We now try to understand these results. First consider $\theta = 0$. It follows from (5.4) that $\theta_1 = \theta_2 = 0$, so, according to (5.3),

$$F(e_1) = \left(\frac{\mu}{v_1} + \frac{1-\mu}{v_2}\right)^{-1}.$$

It is easy to check that $F(\cos\theta,\sin\theta)\cos\theta \leqslant F(e_1)$, and thus

$$\bar{v}_n(e_1) = \left(\frac{\mu}{v_1} + \frac{1-\mu}{v_2}\right)^{-1}. \tag{5.6}$$

We can understand this by noting that any vertical line will translate in the e_1-direction with velocity alternating between v_1 and v_2, so that its effective velocity is their harmonic mean, in agreement with (5.6). Furthermore, any front initially between two vertical lines will remain trapped between them for good and will be forced to move in the e_1-direction with the same average velocity by the comparison principle for Hamilton–Jacobi equations.

Now turn to the case $\theta = \frac{1}{2}\pi$, where the effective normal velocity is given by (5.5). We can again understand this through an explicit construction. Let the initial front be the graph of the function

$$g(x_1) = \begin{cases} 0 & \text{if } x_1 \in [0,\mu],\\ (\mu - x_1)\cot\phi & \text{if } x_1 \in [\mu, \frac{1}{2}(1+\mu)],\\ (x_1 - 1)\cot\phi & \text{if } x_1 \in [\frac{1}{2}(1+\mu), 1],\end{cases}$$

and extended periodically with $\phi \in (0, \frac{1}{2}\pi)$ determined by

$$\sin\phi = \frac{v_2}{v_1}. \tag{5.7}$$

This is shown in figure 4*b*. This initial front has an average normal e_2 (since $g(0) = g(1)$), and propagates without changing shape in the e_2-direction with constant speed v_1. Thus the effective normal velocity is equal to v_1, in agreement with (5.5).

This construction can be modified for θ less than, but close to, $\frac{1}{2}\pi$. Consider as an initial front the graph of the function

$$g(x_1) = \begin{cases} 0 & \text{if } x_1 \in [0, \mu], \\ (\mu - x_1)\cot\phi & \text{if } x_1 \in [\mu, 1-l], \\ (x_1 - 1)\cot\phi - \cot\theta & \text{if } x_1 \in [1-l, 1], \end{cases}$$

with ϕ still given by (5.7) and

$$l = \frac{1}{2}\left(1 - \mu - \frac{v_2}{\sqrt{v_1^2 - v_2^2}}\cot\theta\right).$$

This front has an average normal $\lambda = (\cos\theta, \sin\theta)$ and propagates without changing shape in the e_2-direction with constant velocity v_1. Thus the effective normal velocity (the velocity projected in the direction of the average normal) is equal to $v_1 \sin\theta$. This relationship is shown in figure 5 as the dot–dash line. This construction remains possible as long as $l > 0$, or

$$\cot\theta < (1-\mu)\sqrt{\frac{v_1^2}{v_2^2} - 1}. \tag{5.8}$$

This angle is indicated in figure 5 using the dotted line.

We conclude with two remarks. First, these constructions may be easily generalized to higher dimensions. And second, even though the normal velocity law is isotropic at each point, homogenization gives rise to an anisotropic medium. This stays true for the checkerboard, as we see in the following example.

Example 5.5. Consider a checkerboard composite with the normal velocity law

$$f(x) = \begin{cases} v_1 & \text{if } x \in ([0, \frac{1}{2}] \times [\frac{1}{2}, 1]) \cup ([\frac{1}{2}, 1] \times [0, \frac{1}{2}]), \\ v_2 & \text{otherwise,} \end{cases}$$

extended by periodicity to all $\mathbb{R}^2$, where $v_1 > v_2 > 0$. Note that the boundaries of the squares of the checkerboard have the higher velocity. Following an example worked out by Acerbi and Buttazzo in [1], we also assume $v_1 \geqslant 2v_2$, so that we can deduce an explicit formula for the effective normal velocity.

As in the case of the laminate, we first find $F(\lambda)$. We may leave aside the case $\lambda = e_2$, since it can easily be deduced that

$$F(e_2) = v_1. \tag{5.9}$$

As before, we choose a sequence $D_n = n/|\lambda_1|$ and write

$$F(\lambda) = \lim_{n\to\infty} \frac{D_n}{\inf_{\gamma \in \mathcal{D}^{0,\lambda}_{D_n}} T(\gamma)}. \tag{5.10}$$

Since f is piecewise constant, we can also argue as before that the infimum above is attained on paths that are piecewise linear. Let γ_n be the optimal path.

We now argue that we may, in fact, choose γ_n to be contained in the region where $f = v_1$. Assume that γ_n contains segments in the region $f = v_2$. Since regions with

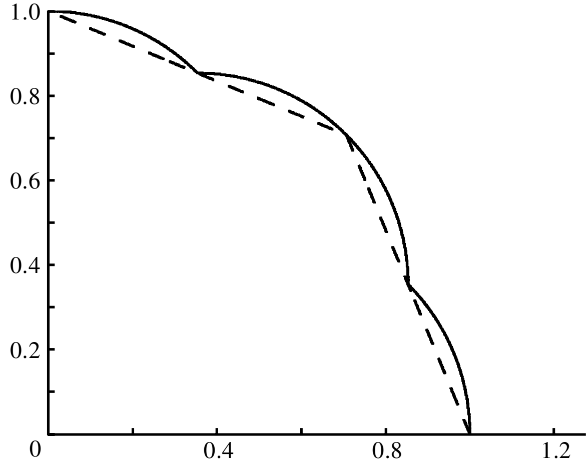

Figure 6. Polar plot of the effective normal velocity for the checkerboard example is shown as the solid curve. The dashed curve shows the polar plot of F.

$f = v_2$ are isolated squares of edge $\frac{1}{2}$, we may construct another path $\tilde{\gamma}_n$ that avoids the region $f = v_2$ by skirting around it along the boundary square of edge $\frac{1}{2}$. Since it is possible to choose this roundabout path such that its length is at most twice the length of the straight segment and we have $v_1 \geqslant 2v_2$, we see that $T(\tilde{\gamma}_n) \leqslant T(\gamma_n)$.

Now, any segment of the optimal γ_n that is piecewise linear and contained in the region where $f = v_1$ is vertical with length $\frac{1}{2}$, horizontal with length $\frac{1}{2}$ or diagonal with length $\frac{1}{2}\sqrt{2}$ (with the possible exception of the last one, since $2n\lambda_2/\lambda_1$ does not necessarily have to be an integer). It is then easy to see that the length of γ_n has to satisfy the supplementary inequality

$$\int_{\gamma_n} \mathrm{d}l \geqslant \frac{n}{|\lambda_1|}((\sqrt{2}-1)\min(|\lambda_1|,|\lambda_2|) + \max(|\lambda_1|,|\lambda_2|)), \tag{5.11}$$

and thus

$$F(\lambda) \leqslant \frac{v_1}{(\sqrt{2}-1)\min(|\lambda_1|,|\lambda_2|) + \max(|\lambda_1|,|\lambda_2|)}.$$

On the other hand, if z is the point in $\mathbb{Z}^2$ closest to $2n\lambda_2/\lambda_1$, we may consider the path γ'_n composed of segments

$$[0, \tfrac{1}{2}(\min(z_1,z_2), \min(z_1,z_2))], \quad [\tfrac{1}{2}(\min(z_1,z_2), \min(z_1,z_2)), \tfrac{1}{2}z], \quad [\tfrac{1}{2}z, n\lambda/\lambda_1].$$

We get

$$F(\lambda) \geqslant \lim_{n\to\infty} \frac{D_n}{T(\gamma'_n)} = \frac{v_1}{(\sqrt{2}-1)\min(|\lambda_1|,|\lambda_2|) + \max(|\lambda_1|,|\lambda_2|)}.$$

Putting these together,

$$F(\lambda) = \frac{v_1}{(\sqrt{2}-1)\min(|\lambda_1|,|\lambda_2|) + \max(|\lambda_1|,|\lambda_2|)}.$$

Using (3.4),

$$\bar{v}_n(p) = \sup_{\lambda \in \partial B_1(0)} \frac{v_1 \cos(\lambda, p)}{(\sqrt{2} - 1)\min(|\lambda_1|, |\lambda_2|) + \max(|\lambda_1|, |\lambda_2|)}.$$

Figure 6 shows polar plots of F (dashed line) and $\bar{v}_n$ (solid line) for the case $v_1 = 1$. Notice that these are not constant so that the effective medium is once again anisotropic.

If we drop the restriction $v_1 \geqslant 2v_2$, we may still argue that the resulting medium is anisotropic. However, we do not obtain an explicit relation.

We can generalize these results to the microgeometry, where the squares on which f is constant are replaced with rectangles congruent to $[0, a_1] \times [0, a_2]$. We can show, as before, that if $v_1 \geqslant ((a_1 + a_2)/a_2)v_2$. Then

$$\bar{v}_n(p) = \sup_{\lambda \in \partial B_1(0)} \frac{v_1 \cos(\lambda, p)}{((\sqrt{a_1^2 + a_2^2} - a_1)/a_2)|\lambda_2| + |\lambda_1|},$$

where we have chosen our coordinates such that $|\lambda_1|/a_1 \geqslant |\lambda_2|/a_2$.

We can also generalize to higher dimensions with a 'checkercubes' microgeometry. If $v_1 \geqslant 2v_2$, we get

$$\bar{v}_n(p) = \sup_{\lambda \in \partial B_1(0)} \frac{v_1 \cos(\lambda, p)}{|\lambda_1| + \sum_{i=2}^{N}(\sqrt{i} - \sqrt{i-1})|\lambda_i|},$$

where the coordinates are numbered such that $|\lambda_i| \leqslant |\lambda_j|$ for $i > j$.

EXAMPLE 5.6. The packed-square microgeometry suggested by Concordel in [6] is similar. Consider a normal velocity defined on $\mathbb{R}^N$ as

$$f(x) = \begin{cases} v_1 & \exists 1 \leqslant i \leqslant N \text{ such that } x_j \in \mathbb{Z}\ \forall j \neq i, \\ v_2 & \text{otherwise,} \end{cases}$$

with $v_1 \geqslant 2v_2 > 0$. Intuitively, this places a grid (obtained by periodically repeating the coordinate planes) with fast normal velocity v_1 in a space with slow normal velocity v_2.

For $\lambda_1 \neq 0$, any minimizer $\gamma_n \in \mathcal{D}_{D_n}^{0,\lambda}$ (where $D_n = n/|\lambda_1|$) is again piecewise linear, and the inequality satisfied by v_1 and v_2 prevents it from entering the slower-speed regions. So γ_n is composed only of unit length segments parallel with one of the coordinate directions e_i (with the possible exception of the last segment, which may have any length). Thus γ_n satisfies

$$\int_{\gamma_n} \mathrm{d}l \geqslant \frac{n}{|\lambda_1|} \sum_{i=1}^{N} |\lambda_i|.$$

Constructions analogous to the $\tilde{\gamma}_n$ from the previous example hold and we get

$$\bar{v}_n(p) = \sup_{\lambda \in \partial B_1(0)} \frac{v_1 \cos(\lambda, p)}{\sum_{i=1}^{N} |\lambda_i|}. \tag{5.12}$$

Figure 7 shows the polar plots of F (dashed line) and $\bar{v}_n$ (solid line) for the case $v_1 = 1$.

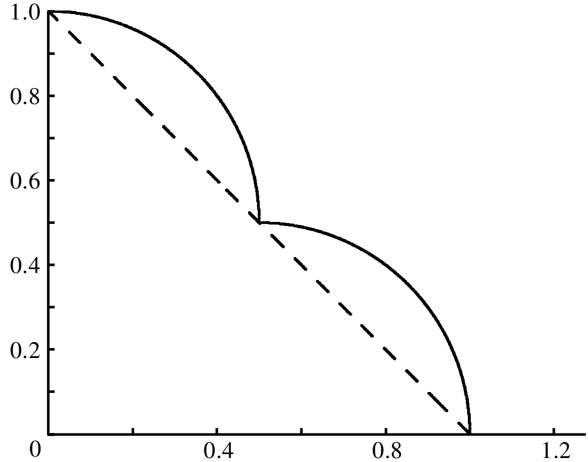

Figure 7. Polar plot of the effective normal velocity for the packed-squares example is shown as the solid curve. The dashed curve shows the polar plot of F.

If the unit cube is replaced with the parallelepipeds

$$\prod_{i=1}^{N}[0,a_i] \quad \text{and} \quad v_1 \geqslant \frac{a+b}{a}v_2,$$

where a and b ($a \leqslant b$) are the smallest two edges of the parallelepiped, equation (5.12) still holds. Furthermore, it can be proved that the resulting medium is anisotropic even if we drop the requirement $v_1 \geqslant ((a+b)/a)v_2$.

Finally, this example may be easily generalized to the case when $f(x)$ equals v_1 whenever k of the coordinates of x are integer (for some k between 1 and $N-1$) and v_2 otherwise.

Example 5.7. We conclude by consider an example where the microgeometry consists of a square packing of slow discs in a fast medium. Consider a normal velocity law defined on $[0,1)^2$ by

$$f(x) = \begin{cases} v_2 & \text{if } (x_1-\frac{1}{2})^2 + (x_2-\frac{1}{2})^2 < \frac{1}{4}, \\ v_1 & \text{otherwise,} \end{cases}$$

extended by periodicity to all $\mathbb{R}^2$, with $v_1 \geqslant \frac{1}{2}\pi v_2 > 0$.

We argue as before that the optimal paths $\gamma_n \in \mathcal{D}^{0,\lambda}_{D_n}$ (where $D_n = n/|\lambda_1|$) are piecewise constant and are prevented from entering the slow discs in view of the inequality satisfied by v_1 and v_2. Therefore, γ_n mainly consists of horizontal and vertical segments of unit length and of quarter circles. Thus γ_n has to satisfy

$$\int_{\gamma_n} \mathrm{d}l \geqslant \frac{n}{|\lambda_1|}(\min(|\lambda_1|,|\lambda_2|)(\tfrac{1}{2}\pi - 1) + \max(|\lambda_1|,|\lambda_2|)).$$

Constructions analogous to the $\tilde{\gamma}_n$ from example 5.5 hold and we get

$$\bar{v}_n(p) = \sup_{\lambda \in \partial B_1(0)} \frac{v_1 \cos(\lambda, p)}{\min(|\lambda_1|,|\lambda_2|)(\frac{1}{2}\pi - 1) + \max(|\lambda_1|,|\lambda_2|)}.$$

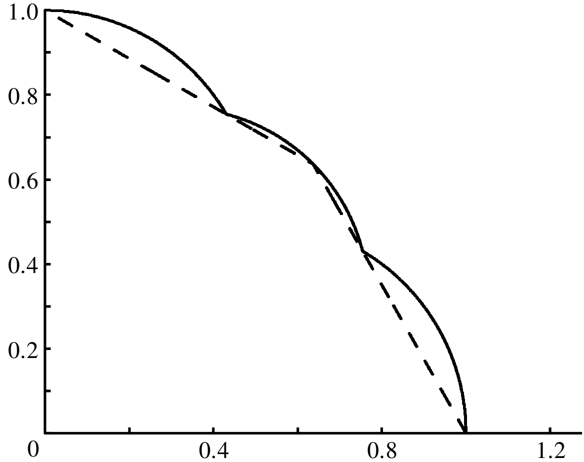

Figure 8. Polar plot of the effective normal velocity for the packed-discs example is shown as the solid curve. The dashed curve shows the polar plot of F.

Figure 8 shows the polar plots of F (dashed line) and $\bar{v}_n$ (solid line) for the case $v_1 = 1$. The resulting medium is clearly anisotropic.

This example may easily be generalized to the microgeometry, where, instead of a slower-speed circle inscribed in the unit square, we have a slower-speed ellipse inscribed in the rectangle $[0, a] \times [0, b]$. It can also be proved that the resulting homogenized medium is anisotropic even if we drop the requirement $v_1 \geqslant \frac{1}{2}\pi v_2$.

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